Eigenvectors and Eigenvalues.

Main question:

- Let $T: V \to V$ be a linear operator on a (finite dimensional) vector space.
- Let $\mathcal{B} = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\}$ be a basis for V.
- The matrix associated with T with respect to $\mathcal{B}$ is given by

$$M_T^{\mathcal{B}} = \left([T(\bar{b}_1)]_{\mathcal{B}} \cdots [T(\bar{b}_n)]_{\mathcal{B}} \right)$$

Is it possible to find some basis C of V such that M_T^C is as simple as possible?

Diagonal

Recall that:

$$M_T^C = \underset{C \leftarrow B}{P} \, M_T^B \, \underset{B \leftarrow C}{P}$$

Goal: Find a basis C (when possible) such that M_T^C is diagonal

$T: V \to V$

$$M_T^C = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}_{n \times n}$$

$\dim V = n$

That is, if $C = \{\bar{c}_1, \bar{c}_2, \ldots, \bar{c}_n\}$

$$M_T^C = \left([T(\bar{c}_1)]_C \; [T(\bar{c}_2)]_C \cdots [T(\bar{c}_n)]_C \right)$$

$$= \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

- $[T(\bar{c}_1)]_C = \begin{pmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow T(\bar{c}_1) = \lambda_1 \bar{c}_1$

$[T(\bar{c}_2)]_C = \begin{pmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \Rightarrow T(\bar{c}_2) = \lambda_2 \bar{c}_2$

$\vdots$

$[T(\bar{c}_n)]_C = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \lambda_n \end{pmatrix} \Rightarrow T(\bar{c}_n) = \lambda_n \bar{c}_n$

Eigenvalues and Eigenvectors.

Definition. Let $T: V \to V$ be a linear mapping.

- An eigenvector of T is a non zero vector $\bar{x} \in V$ such that

$$T(\bar{x}) = \lambda \bar{x} \qquad \bar{x} \neq \bar{0}$$

for some scalar λ.

eigenvector:
- vector propio
- autovector

- The scalar λ is called the eigenvalue associated with $\bar{x}$.

$$\begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & 0 \\ & & \lambda_3 \end{pmatrix}$$

Eigenvalues are allowed to be zero.

Example. Let $T : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by

$\mathbb{C}^2 = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : x, y \in \mathbb{C} \right\}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ x \end{bmatrix}.$$

Eigenvalues? $\lambda_1 = i$, $\lambda_2 = -i$

$\lambda_1 = i$ $\quad T\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = i\begin{bmatrix} x \\ y \end{bmatrix}$

$\boxed{-y = ix} \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \qquad \alpha \in \mathbb{R}$

$i \to x = iy$

$ix = -y \qquad -y = i(iy) = -y \Rightarrow 0 = 0$

$\lambda_2 = -i$ if $\begin{bmatrix} x \\ y \end{bmatrix}$ is an eigenvector then

$T\begin{bmatrix} x \\ y \end{bmatrix} = -i\begin{bmatrix} x \\ y \end{bmatrix} \qquad \begin{bmatrix} -y \\ x \end{bmatrix} = -i\begin{bmatrix} x \\ y \end{bmatrix}$

$-y = -ix$ $\Rightarrow$ $\begin{bmatrix} x \\ y \end{bmatrix} = \beta \begin{bmatrix} -i \\ 1 \end{bmatrix}$

$x = -iy \leftarrow \cdot -i$

$-ix = +i(+i)y = -y$

$x = -iy = -i\beta$

$y = \beta$

Basis of $\mathbb{C}^2$ formed totally by eigenvectors of T: $\mathcal{C} = \left\{ \begin{pmatrix} i \\ 1 \end{pmatrix}, \begin{pmatrix} -i \\ 1 \end{pmatrix} \right\}$

$M_T^{\mathcal{C}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$

$\dim \mathbb{C}^2 = 2$. 2 dinctinct eigenvalue

$\dim V = 5$, 3 distinct eigenvalues

$$\begin{array}{ccc} & \lambda & & \mu \\ \bar{v}_1 & & \bar{v}_2 & \bar{v}_3 \end{array}$$

$\{\bar{v}_1, \bar{v}_3\}$ l. ind.

$\{\bar{v}_2, \bar{v}_3\}$. l. ind.

$\{\bar{v}_1, \bar{v}_2\}$?

linear:

- function
- mapping.
- transformation
- operator.

Theorem. Let $T: V \to V$ be a linear operator. Let $\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m$ be eigenvectors associated with distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_m$ ($\lambda_i \neq \lambda_j$ when $i \neq j$). Then $\{\bar{x}_1, \bar{x}_2, \dots, \bar{x}_m\}$ is a linearly independent set.

Corolary Let $T: V \to V$ be a linear operator. Then T has at most $\dim V$ eigenvectors.

$\dim V = n$

lin. ind.

$$\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}_{n \times n}$$

at most n distinct eigenvalues

$$\Big\Downarrow \text{Theorem}$$

at most n lin. ind. eigenvectors

Example Let V be the (infinite dimensional) vector space of functions with derivatives of any order.

Let $T: V \to V$ be a linear operator defined by $T[f] = f'$.
For each $\lambda \in \mathbb{R}$, consider the function
$f_\lambda(t) = e^{\lambda t}$. $\leftarrow$ eigenvectors of T

$$T[e^{\lambda t}] = \frac{d}{dt} e^{\lambda t} = \lambda e^{\lambda t} \qquad T[f_\lambda] = \lambda f_\lambda$$

$$\lambda_1 \neq \lambda_2 \qquad \{e^{\lambda_1 t}, e^{\lambda_2 t}\} \text{ lin ind.}$$

Example. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator defined by

$$T(\bar{x}) = \underbrace{\begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}}_{M_T^{\mathcal{E}_3,\mathcal{E}_3}} \bar{x}$$

$2, 9$

- $\lambda_1 = 2$ is an eigenvalue of T with associated eigenvectors

$$\underbrace{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}, \underbrace{\begin{pmatrix} -2 \\ 2 \\ 1 \end{pmatrix}}_{\begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}}$$

- $\lambda_2 = 9$ is an eigenvalue of T with associated eigenvector $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

λ: $T(\bar{x}) = \lambda \bar{x}$
$T(\bar{y}) = \lambda \bar{y}$

$$T(\alpha \bar{x} + \beta \bar{y}) = \alpha T(\bar{x}) + \beta T(\bar{y})$$
$$= \alpha \lambda \bar{x} + \beta \lambda \bar{y}$$
$$T(\alpha \bar{x} + \beta \bar{y}) = \lambda(\alpha \bar{x} + \beta \bar{y})$$

$$T\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}4&-1&6\\2&1&6\\2&-1&8\end{pmatrix}\begin{pmatrix}1\\2\\0\end{pmatrix}=\begin{pmatrix}2\\4\\0\end{pmatrix}=2\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$T\begin{pmatrix}-3\\0\\1\end{pmatrix}=\begin{pmatrix}4&-1&6\\2&1&6\\2&-1&8\end{pmatrix}\begin{pmatrix}-3\\0\\1\end{pmatrix}=\begin{pmatrix}-6\\0\\2\end{pmatrix}=2\begin{pmatrix}-3\\0\\1\end{pmatrix}$$

$$E_2=\operatorname{Span}\left\{\begin{pmatrix}1\\2\\0\end{pmatrix},\begin{pmatrix}-3\\0\\1\end{pmatrix}\right\}$$

$$E_9=\operatorname{Span}\left\{\begin{pmatrix}1\\1\\1\end{pmatrix}\right\}$$

Definition. Let $T: V \to V$ be a linear operator. For each eigenvalue λ of T, we define the set

$$E_\lambda = \text{Span}\{\bar{x} : T(\bar{x}) = \lambda \bar{x}\} \subseteq V$$

Called the eigenspaces associated with λ.

Theorem E_λ is a subspace of V.

We call E_λ the <u>eigenspace</u> associated with λ.

Theorem* Let $T: V \to V$ be a linear operator, at let M_T^B be its associated matrix with respect to a basis B

For each eigenvalue λ of T: $T(\bar{x}) = A\bar{x}$

$T(\bar{x}) = \lambda \bar{x}$

$T(\bar{x}) - \lambda \bar{x} = \bar{0}$

$(T - \lambda I_d)(\bar{x}) = \bar{0}$

- $E_\lambda = \mathrm{Ker}(T - \lambda I_d)$
  $(\mathrm{Nul}(M_T^B - \lambda I_n))$
- $\dim E_\lambda = \dim V - \mathrm{Rank}(M_T^B - \lambda I_n)$
- λ is a root of the polynomial
  $$\det(M_T^B - \lambda I_n)$$
  degree $\dim V$

$M_T^B [\bar{x}]_B - \lambda [\bar{x}]_B = \bar{0}$

$(M_T^B - \lambda I) [\bar{x}]_B = 0$

$$\overbrace{T(\bar{x}) = \lambda \bar{x}} \xrightarrow{B} [T(\bar{x})]_B = \overbrace{M_T^B [\bar{x}]_B = \lambda [\bar{x}]_B}$$

B

Definition. Let $T: V \to V$ be a linear operator and let $M_T^{\mathcal{B}}$ be its associated matrix with respect to a basis $\mathcal{B}$.

- The polynomial

$$\det(M_T^{\mathcal{B}} - \lambda I_n)$$

of degree $n = \dim V$ is called the characteristic polynomial of T (or of $M_T^{\mathcal{B}}$).

- $\det(M_T^{\mathcal{B}} - \lambda I_n) = 0$ is called the characteristic equation of T (or of $M_T^{\mathcal{B}}$).

Example Find the eigenvalues and eigenvectors of the linear mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 2 & 3 \\ 3 & -6 \end{pmatrix}$$

Example Find the eigenvalues and eigenvectors of the linear mapping $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined by $T(\bar{x}) = A\bar{x}$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Example Find the eigenvalues and eigenvectors of the linear mapping $T: \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 5 & -2 & 6 & -1 \\ 0 & 3 & -8 & 0 \\ 0 & 0 & 5 & 4 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Theorem. The characteristic polynomial of a linear operator is invariant under change of basis.

Multiplicity of eigenvalues

Example. $A = \begin{pmatrix} 2 & 2 & 3 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}$

Example $A = \begin{pmatrix} 2 & 2 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 2 \end{pmatrix}$

Theorem Let $T: V \to V$ be a linear operator and let $p(x)$ be its characteristic polynomial.

For each eigenvalue λ,

$$p(x) = (x-\lambda)^{m_\lambda} q(x), \quad q(\lambda) \neq 0$$

and $1 \leq \dim E_\lambda \leq m_\lambda$ where E_λ is the eigenspace associated with λ.

Diagonalizable linear mappings

Definition. A linear mapping $T: V \to V$ is diaganolizable if there exists a basis

$$B = \{\bar{b}_1, \bar{b}_2, \dots, \bar{b}_n\}$$

of V made up entirely of eigenvectors of T. Consequently,

$$M_T^{B,B} = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix}$$

where $T(\bar{b}_i) = \lambda_i \bar{b}_i$, $i = 1, 2, \dots, n$.

Theorem. A linear mapping $T: V \to V$ ($n = \dim V$) is diagonalizable if and only if T has n linearly independent eigenvectors.

Corollary. If $\lambda_1, \lambda_2, \ldots, \lambda_k$ are distinct eigenvalues of a linear mapping $T: V \to V$, then T is diagonalizable if and only if

$$\dim E_{\lambda_1} + \dim E_{\lambda_2} + \cdots + \dim E_{\lambda_k} = n.$$

- If and only if, for each eigenvalue λ with multiplicity m_λ, $\dim E_\lambda = m_\lambda$.

- If and only if, for each eigenvalue λ with multiplicity m_λ, there exist m_λ linearly independent eigenvectors associated with λ.

Corollary. If each eigenvalue of $T: V \to V$ has multiplicity 1, then T is diagonalizable.

Example If possible, diagonalize the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

Example If possible, diagonalize the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where

$$A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$$

Example If possible, diagonalize the the linear mapping $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(\bar{x}) = A\bar{x}$ where $A = \begin{pmatrix} 5 & -8 & 1 \\ 0 & 0 & 7 \\ 0 & 0 & -2 \end{pmatrix}$